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Student Perspectives on Equation: The Transition from School to University

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It has been recognised that many student perspectives on equations and their use of the equals sign have not mirrored those that mathematicians would like to see in tertiary students. This paper tracks transition of understanding of the equals sign by comparing secondary school students' thinking with that of first year university students. We analyse the understanding displayed in terms of properties of the constituent parts of equations, identifying a number of incomplete or pseudo-conceptions that are sometimes influenced by representational aspects of the properties, and other times by apparent over-generalisation of a property. A start is made on constructing a framework for understanding of the mathematical equation object that could assist in the transition from school to tertiary mathematics study.

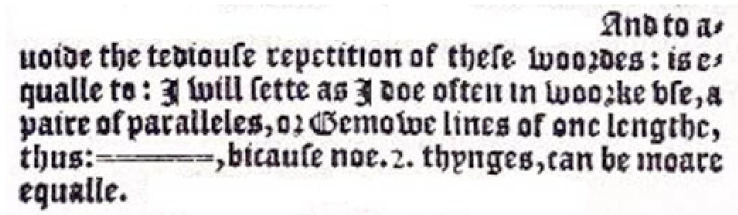
Background

Ubiquitous mathematical concepts such as equation, where understanding forms a crucial part of the mathematical experience from early school years right through to tertiary study, need to form part of any discussion of the transition from school to university. While to the experienced mathematical eye equations appear as a single object they are often seen by students to consist of a number of separate entities. Each of these parts, and indeed the gestalt they comprise, may, according to Laborde (2002), be viewed from several perspectives including a surface or perceptual one, and a mathematical one, from which the mathematical properties of the entity or object are understood. In this paper we seek to examine the role of the mathematical properties that constitute the concept of equation, by reference to the embodied, symbolic and formal worlds of thinking (Tall, 2004, 2007). This involves a consideration of arithmetic numbers, symbolic literals, operators, the '=' symbol itself, and the formal equivalence relation, and how each may contribute to understanding of equation for students at different stages of mathematical development. Hence, our hypothesis is that understanding the mathematical equation object requires the formation and integration of individual properties from a number of areas, and that the crucial binding agent for such understanding of the constituent parts is language, although this aspect is not explicitly addressed here.

There is no doubt that many students struggle to attach meaning to many of the symbols used in mathematics. Mason (1987) suggests that a semiotic problem, concerning the relationship between the sign and the signified, or the symbol and the symbolised, is at the root of this. For equation, the use of the sign '=' to signify 'is equal to' dates back to when Robert Recorde in the Whetstone of Witte (1557) first used '=====' (see Figure 1 for the original and a translation into modern English). Prior to this time mathematicians laboriously wrote out the words '…is equal to…', which was sometimes abbreviated to ae (or oe), from the Latin for equal—aequalis (Lacey, 2004).

Further, the process of attaching appropriate meaning to mathematical symbols may be subverted by teaching that is heavily weighted in favour of

instrumental learning (Skemp, 1976). Such a learning environment encourages a process-oriented view of mathematics (Thomas, 1994), where the object of study is not cognitively engaged, and hence pseudo-conceptions (Vinner, 1997) are more likely to occur. Once these pseudo-conceptions are in place they can be very resistant to change and may act as cognitive obstacles when a student is encouraged to perceive a mathematical object, such as an equation, via its properties. For example, while many students develop a reasonable working knowledge of arithmetic numbers and their operators, the same cannot be said of symbolic literals (Küchemann, 1981).


And to auoide the tediouse repetition of these woordes : is equalle to : I will sette as I doe often in woorke vse, a paire of paralleles, or Gemowe lines of one lengthe, thus:=====, bicause noe.2. thynges, can be moare equalle.

And to avoid the tedious repetition of these words: is equal to: I will set as I do often in work use, a pair of parallels, or Gemowe [ie twin] lines of one length, thus: =====, because no 2 things, can be more equal.

Figure 1. Robert Recorde's (1557) first use of an '=' sign.

Previous research agrees with placing an emphasis on symbolic understanding, and has suggested that to use equations in a versatile, mathematical way a strong symbol sense should be developed (Arcavi, 1994; Pierce & Stacey, 2001; Warren, 2001), rather than just an input-output process understanding. While symbol sense has not been fully defined, it should include the knowledge that the correctness of an algebraic transformation may be checked, and that modelling particular phenomena requires a particular type of function. Further, a view of letters as encapsulated objects (Tall, Thomas, Davis, Gray, & Simpson, 2000) appears to be an important attribute. However, those who have espouse symbol sense, such as Arcavi (1994), have often limited their discussion to behaviour demonstrating good use of literal symbols, with less emphasis on other symbols, such as the equals sign. Pierce and Stacey's (2001) framework for algebraic insight characterises symbol sense in terms of two key aspects: algebraic expectation, requiring an emphasis on the structure and key features of algebraic expressions, and the ability to link representations.

There is clear evidence that school students exhibit problems going beyond the separator-operator interpretation of the equals symbol (Baroody & Ginsburg, 1983). While some can develop flexibility in accepting the use of the equals symbol in a variety of arithmetic sentence structures (Denmark, Barco and Voran, 1976) they may still view the equals symbol primarily as an operator rather than a relational symbol. Investigating student acceptance of an equivalent amount written in a variety of ways (e.g., 4+7=12–1), Herscovics and Kieran (1980) found that students were able to accept and work comfortably with arithmetic identities containing multiple operations on both sides. This suggests that the meaning of the

equals symbol needs to evolve from the intuitive ideas of sameness, or counting the total found in arithmetic (Gelman & Gallistel, 1978) and the idea of the result of or answer to a procedure (Kieran, 1981), to a notion of the equivalence of algebraic statements with reflexive, symmetric and transitive properties. While this process of change does not appear to come easily or quickly to many students, Whitman and Okazaki's (2003) results showed that student understanding of "=" as equivalence could be improved from the first to second grade. In her research examining the ideas of quantitative sameness with 5 year olds Warren (2007) also concluded that, not only are they capable of this conception, but they can represent it using real world contexts and symbolic form.

In the light of the above, when we ask the question 'What is an equation?' we may get a number of differing responses, with schools and universities providing different emphases. Tall's theory of three worlds of mathematical thinking provides a framework for looking at how and why a teaching sequence influences the growth of the equation as a mathematical object for the learner. Tall (2004, 2007), describes three distinct but interrelated worlds of mathematical thinking, and bases his theoretical framework on a consideration of the different types of activity that underpin the mathematical thinking in each world. This model acknowledges that children perceive and act on the world, and build up increasingly sophisticated mathematical ideas by reflecting on the results of their perceptions and actions. This occurs in two different ways. If perception is the focus of thought about real world objects a categorization process takes place leading to the development of properties and a more sophisticated conception through performing thought experiments. This world, incorporating the iconic and enactive modes of Bruner (1966), he has named the *embodied world*. Applying this idea to equations, we see that thinking about relationships in the embodied world gives the child an initial experience with equations, an opportunity to grapple with the ideas of equality, without the need for any sort of written symbolism. However, at some stage they will need to shift their "…attention from the steps of an action to the effect and imagining the effect as an embodied object" (Tall, 2007, p. 5), thus moving to a stage of embodied compression.

The introduction of symbols allows us to carry out processes e.g. addition, differentiation, but also allow us to think of the symbols as concepts that may be held in the mind e.g. sum, derivative. This duality of symbol use is termed a *procept* (Gray & Tall, 1994), and enables precise quantitative information to be obtained beyond that possible in an embodied world. This world is termed the *symbolic world*. A change from symbolic manipulation to a formal approach in which definitions are formulated to build a systematic axiomatic theory heralds the move from the symbolic world to the *formal world*, the third of Tall's worlds of mathematical thinking.

One view of an equation is as a structural statement, or representation of a mathematical relationship between entities that are the objects of an algebraic and/or arithmetic representation system. One of the requirements for generating and adequately interpreting an equation structurally is a conception of equivalence. In the symbolic world this is usually achieved by algebraic manipulation leading to perception of the same structural form. However, sometimes there is no need for such manipulation, and in their research with preservice teachers, Hansson and Grevholm (2003) found that very few preservice teachers considered that $y=x+5$ was an equation, instead tending to a numerical interpretation of $y=x+5$. As thinking progresses in the transition from school to university, from the symbolic world to the formal world, the reflexive, symmetric,

and transitive character of the equals sign should come to the fore, and it is through these properties, recast as axioms, that the formal world conveys the concept of an equivalence relation. On this topic of equivalence, Gattegno (1974, p. 83), stated:

> We can see that identity is a very restrictive kind of relationship concerned with actual sameness, that equality points at an attribute which does not change, and that equivalence is concerned with a wider relationship where one agrees that for certain purposes it is possible to replace one item by another. Equivalence being the most comprehensive relationship it will also be the most flexible, and therefore the most useful.

In spite of these ideas it is not that easy to specify precisely what an equation is, or may be. A Collins mathematics dictionary (Borowski & Borwein, 1989, p. 194) deals with the definition in this way

> Equation, *n*. a formula that asserts that two expressions have the same value; it is either an *identical equation* (usually called an IDENTITY), which is true for any values of the variables, or a *conditional equation*, which is only true for certain values of the variables (the ROOTS of the equation).

Thus they specify two possible types of equation: a *conditional equation;* and an *identical equation* and so, for example, $2x+1=6$ would be a *conditional equation*, but $2(2x+1)=4x+2$ would be an *identical equation* (or equivalence); we make use of this distinction in our analysis below.

The purpose of this present research study was to identify the progressive nature of student thinking about distinctive properties of the different types of equation during the transition from secondary school to university, in order to identify areas where targeted teaching might improve construction of the mathematical object. Thus we tracked students' perceptions of what constitutes an equation and related these to the mathematical object.

Method

The results discussed here are taken from a large cross-sectional study aimed at mapping out changes in student use and understanding of the equals sign through secondary school and into tertiary study. We present results from three groups of students, in the lower secondary school, upper secondary school, and first year at university.

The first group (P) comprised 29 Year 10 students (age 14-15 years), 8 female and 21 male, from a large, coeducational, high socio-economic school (decile 10 of 10) in Auckland. They were given a questionnaire with 12 questions aimed at different aspects of equation, previously identified as important to understanding (Godfrey & Thomas, 2003) and appropriate to their age. They were given 55 minutes to complete the questions, doing so in a normal mathematics class.

The second group (Q) contained 76 Year 13 students (16-18 years old), 39 female and 37 male, from two large, multicultural, coeducational, public schools situated in the suburbs of Auckland, but different from the group P school. While these students were from Year 13, the data was collected in late April so the students had had very little experience at this level. School A has a socio-economic rating of decile 6 and school B 4. Both these classes were considered mixed ability by their respective schools, and the students had chosen to study Mathematics with Calculus or Mathematics with Statistics in their final year at school. A questionnaire containing two parts, part A with 6 questions and part B with 5

questions, was given to the students to complete during one of their normal mathematics periods. The first-named researcher supervised the administration of the questionnaire by the teachers, who were keen to be involved.

The third group (R) consisted of 30 first year students at The University of Auckland taking the Mathematical Modelling paper offered by the Engineering Science Department of the Faculty of Engineering. Five of the students (three females and two males) were from the Faculty of Science, doing the Food Science degree programme and the rest of the students (eight females and seventeen males) were from the Faculty of Engineering. The latter group are considered to be of very good mathematical ability, while the former group are generally considered not as good at mathematics. A questionnaire containing 2 parts, 6 questions in the first and 5 questions in the second, was given to the students to complete after one of their normal mathematical modelling lectures and the first-named researcher supervised administration of the questionnaire. They were given 1 hour to complete the questionnaire, which proved to be ample time.

Results

Group P

Figure 2 contains a summary of the 2 of the 12 questions given to Group P that are considered in the analysis below. Question 5 of this questionnaire asked students to identify from a list of five statements those that they thought were equations, giving reasons for their choices.

5. Pick out those statements that are equations from the following list and write down why you think the statement is an equation.

a) $k = 5$

b) $7w - w$

c) $5t - t = 4t$

d) $5r - 1 = -11$

e) $3w = 7w - 4w$

10. If $p = q + 3$ and $q + 3 = 2 - r$, write p in terms of r.

Figure 2. Two of the questions from the group P questionnaire.

Overall the responses to what constitutes an equation for Group P students showed that 10 (34.4)% of them thought that $k = 5$ was an equation, 5 (17.2%) $7w - w$, 21 (72.4%) $5t - t = 4t$, 23 (79.3%) $5r - 1 = -11$, and 24 (82.8%) $3w = 7w - 4w$, as seen in Table 1. The low rate of acceptance of $k = 5$ was not surprising, but we wanted to try and see what the students' thinking was for rejecting this, and the items in the other question parts. A further examination of the individual student responses found that most of them fell into one of the three distinct categories described in Table 2. These categories may be exemplified by the responses of the students shown in Figures 3–6.

Table 1
Group P Questionnaire Responses for Question 1.5 (N=29)

Question 1.5	Yes	No	No Response
a) $k = 5$	10	17	2
b) $7w - w$	5	22	2
c) $5t - t = 4t$	21	6	2
d) $5r - 1 = -11$	23	4	2
e) $3w = 7w - 4w$	24	3	2

Table 2
Categories of Responses for Question 5

Statements that are an equation	Stated reason for the choice	Number of students in category
a, c, d, e	Needs an = sign	8
b, c, d, e	Needs an operation to carry out	3
c, d, e	Needs an = sign and an operation to carry out	9

It seems that the 8 category 1 students (see Figure 3) were basing their decision primarily on the surface structure of the equation; if it contains an equals sign then it is an equation. They responded, for example:

P7. An equation has an = sign in it

P12. Equation, = is present

P13. Because they have equals in them

P21. These are equations because they have an = sign in the statement

P7 P21

Figure 3. Examples of category 1 equation responses in question 5.

In contrast the 3 category 2 students have the perspective that ability to carry out an operation to produce a result is the crucial factor for it to be an equation, regardless of whether there is an explicit '=' sign present. Hence, we see that these students were happy to accept $7w - w$ as an equation, but reject $k = 5$. For the former they no doubt think that the sign is implicit and may be supplied prior to writing the answer (Kieran, 1981). As primarily symbolic world thinkers they place an emphasis on symbolic manipulation. These students explained their thinking in

the following ways (see also Figure 4):

> P1. …they involve taking 2 or more sets of numbers and either subtracting, adding or multiplying or dividing to get another number.
>
> P16. because you have to subtract, add, multiply and/or divide.
>
> P20. because there is still stuff to figure out.
>
> P26. because it envolves [sic] –, +, ×, ÷sign in it.

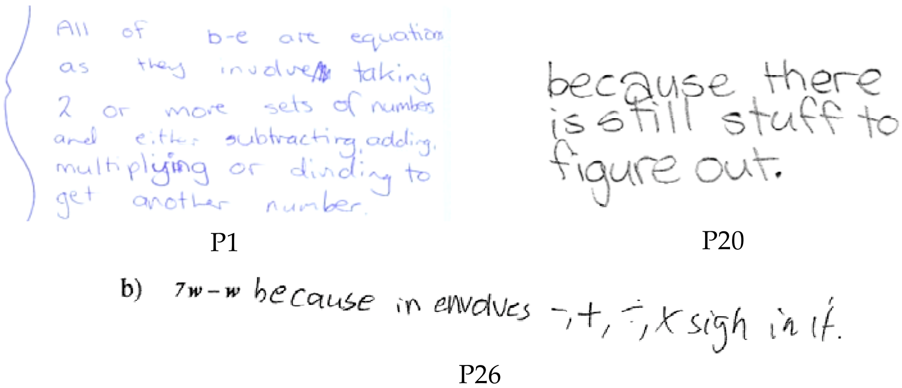


Figure 4. Examples of category 2 equation responses in question 5.

The 9 category 3 students have a subtly different perspective from the second. While there must still be an operator, or as student 8 puts it “more than just one letter or number on the side”, implying an operator between them (see Figure 5), they also require an explicit ‘=’ sign to be present, and hence they reject both $7w - w$ and $k = 5$, for different reasons. They comment that:

> P8. because they have equals sign and more than just one letter or number on the side.
>
> P16. Because it has an answer and a means of getting the answer.
>
> P24. Because it has an answer and uses subtraction.
>
> P29. They all have an = sign and it’s not just a statement like a).

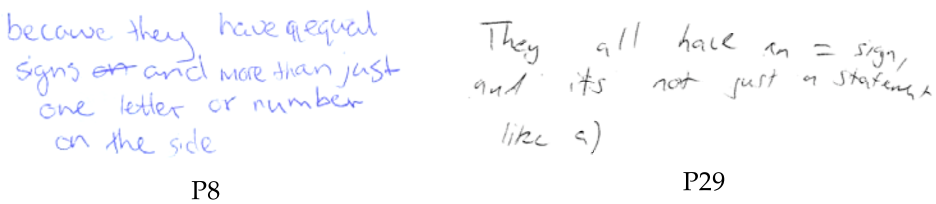


Figure 5. Examples of category 3 equation responses in question 5.

Among those not fitting neatly into this threefold classification were 6 students (there were also 3 no-response students) who gave a mixture of answers with little discernible consistency or pattern. It seems that they may be in transition between the groupings we have identified, or they may have developed pseudo-conceptions. What they seem to have in common though is an emphasis on symbolic world thinking and the need for procedural operators, and on wanting to 'solve' an equation. Some of their reasons for answers were:

> P3 – gave c and e, "Because it has a correct answer"; "the answer is right."
>
> P4 – gave b, d and e, "yes, because w stands for a number and you are minusing it from $7w$"; For b he said "no, the t's do not represent a number…", so he has a specific unknown view, but only of particular letters.
>
> P20 – gave b, c and e "because there is still stuff to figure out." She is close to a group 2 member but she only applies the operator to letters and not to part d which has $5r - 1$.
>
> P27 – gave c and d, saying "has equal sign and only one unknown number" stating for e that "it is already true so it is not an equation." He seems to be moving toward category one, but also wants to have some work to do. The reason for his failure to group e with c is shown in Figure 6, where he says that it is 'already true'. This shows that his thinking is in transition.

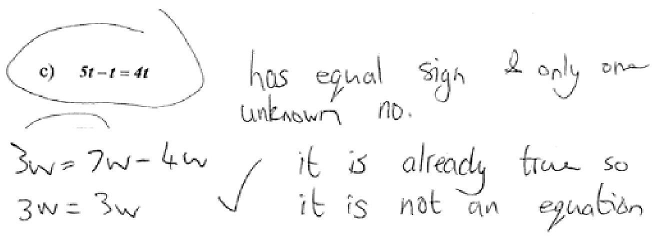


Figure 6. Student P27's equation responses in question 5.

Student P6 chose only 5c) as an equation (see Figure 7), rejecting $3w = 7w - 4w$, and hence was requiring equivalence, but apparently only in the process-oriented, left to right, format identified by Thomas (1994), since she wants the 'sum' to appear before the answer, writing that this 'goes in correct order' and to be correct.

Figure 7. Student 6's equation response in question 5.

We were interested to know how the categories identified above would mesh with student perspectives on transitivity, and Question 10, If $p = q + 3$ and $q + 3 = 2 - r$, write p in terms of r, requires the ability to use the transitivity property of the '=' sign in an equation. Table 3 shows the Question 10 results of the 20 students in the 3 categories (see Figure 2 for the question).

Table 3
Results on Question 10 for the 3 Categories of Students

Category of Understanding of Equation	Number with Q10 correct
1	7 (out of 8)
2	1 (out of 3)
3	3 (out of 9)

Category 2 was a small sample and only one student got the question correct, making them a less successful group. Overall it appeared that the category 1 students were generally more successful than the category 3 group. However, we note that the students who wrote $p=-r+2$ rather than $p=2-r$ had probably not applied the transitive property directly, but had likely resorted to symbolic manipulation to get answer. Only student 20 of the 6 non-categorised students correctly answered question 10. What do these results tell us? It may be deduced that the students who still had some view of equation as requiring operators and solutions did not perform quite as well as those who used the surface structure of the presence of an '=' sign. This latter group may have subsumed other knowledge of equations, such as structure sense, under this umbrella catch-all, since 62.5% of them were able to apply the transitive property, compared with only 33.3% of the members of the other two, primarily procedure-oriented groups.

Table 4
Results on Question 10 for Category 1 and 3 Students

Category 1	7. $p=2-r$; 12. $p=2-r$; 13. $p=2-r$; 15. $p=-r+2$; 17. $p=2r$; 18. $p=2-r$; 21. $p=2-r$; 30. $q=2-r-3$, $q=-1-r$, $p=-r+2$.
Category 3	2. $p-2=r$; 8. $p=2-r$; 9. $p=2-r$; 11. $p=q-2+r$; 14. $p=2-r$; 23. NR; 24. NR; 25. NR; 29. $r=-q-1$, $q+1=-r$.

Group Q

Figure 8 contains a summary of the 3 Group Q questions that are analysed here. Questions 1 and 2 address the way that students define equation, and Question 5 has been chosen to correspond with that given to Group P.

1. Explain what an equation is. [read question 3 below before answering]

2. Give an example(s) of an equation.

5. Look at the following list. Decide which ones you consider are equations and circle the Y. Circle the N if you think they are not equations. If you would like to comment please do so beside each statement.

a) $a = 5$	Y	N
b) $7w - w$	Y	N
c) $5t - t = 4t$	Y	N
d) $0 = x^2 + 2x - 5$	Y	N
e) $3w = 7w - 4w$	Y	N
f) $a = a$	Y	N

Figure 8. Three questions from the group Q questionnaire (some minor changes).

In this case 5d), $0 = x^2 + 2x - 5$, replaced the equation $5r - 1 = -11$ as an appropriate standard of equation that is solved at the students' level. In addition these older students were also asked whether $a=a$ is an equation, in order to see if they were progressing in formation of the reflexive property of equivalence.

Overall the responses to what constitutes an equation for Group Q students are shown in Table 5. Again we see a wide spread of facilities in the questions, from a sizeable minority, 21 (27.6%) accepting $7w - w$ as an equation with an implied equals, to 72 (91.1%) agreeing with $0 = x^2 + 2x - 5$. On the other hand 43 (56.6%) were unwilling to accept $a = 5$, and 47 (61.8%) did not see $a = a$ as an equation, even though they probably considered it to be true. Thus a majority did not accept as valid the assignment [or definition] format, or the equivalence property uses of the '=' sign that are so common in formal world thinking. Like the younger secondary students, these students were still very much symbolic world thinkers.

Table 5
Group Q Responses for Question 5 (N=76)

Question 5	Yes	No	No Response
a) $a = 5$	31	43	2
b) $7w - w$	21	53	2
c) $5t - t = 4t$	62	12	2
d) $0 = x^2 + 2x - 5$	72	2	2
e) $3w = 7w - 4w$	61	13	2
f) $a = a$	27	47	2

As with group P, most of the students' responses in Question 5 fell into the same three categories, although we were able to identify an additional two categories that also appeared to have some students (see Table 6). These latter two categories may have emerged from the crystallization of some of the ideas of the younger students, who seemed to emphasise the solving aspect of an equation by carrying out an operation, but had previously not done so in a systematic manner. Nine students did not fall neatly into any of these classes.

Table 6
Categories of Responses for Question 5 (N=76)

Statements which are an equation	Stated reason for the choice	Number of students in category
a, c, d, e	Needs an = sign	24
b, c, d, e	Needs an operation to carry out	10
c, d, e	Needs an = sign and an operation to carry out	20
b, d	Needs an operation but is not an identity or an assignment	7
d	Needs an operation and an = sign but is not an identity or an assignment	4

For the extra equation, 5f), presenting the reflexive property, one might expect from Table 3 that only the first group of 24 would agree that $a=a$ is an equation, since it has an '=' sign but no operation to perform. In the event this was close to what happened, with 27 (35.5%) saying it is, and 47 (61.8%) that it is not. This result shows that, even in the final year of secondary school relatively few students have developed an understanding of the reflexive property as belonging to the concept of equality or equivalence.

Responses to Question 1 of the 24 students in category 1 seem to confirm the use of a surface structure view of equation (Laborde, 2002), looking at the equation, rather than through it (Mason, 1995) without overlaying the properties of the object on that surface structure (Thomas, 2006). However, the extent to which they are also using an underlying equation schema with other properties is not clear. Typical responses were:

> Q6. Unknown numbers on each side connected with = sign
>
> Q33. Formula that makes sense with = sign
>
> Q41. One thing is = to another

The only clear requirement for the 10 category 2 students was for an operation to be present, with the equals sign implicit or explicit, but this was often not mentioned in their responses. However, they clearly wanted, as a priority, to be able to 'solve', 'find', 'work out' or 'calculate' an answer:

> Q13. Mathematics problem to be solved to get an answer
>
> Q28. Mathematics question requiring an answer
>
> Q39. Mathematics statement which enables user to work out unknowns
>
> Q50. Mathematics sentence/statement…specific calculation to find something

For the category three students an added requirement, apart from an operation to be present, was the addition of an equals sign as a signal of the answer (Kieran, 1981), although they did not often mention it explicitly. Some who did said:

> Q15. One with = sign and some operations
>
> Q59. Both sides the same, with = sign, sometimes an unknown
>
> Q62. Set of numbers and letter[s], used to solve problems, LHS = RHS

Those in category 4, who did not allow their 'solution' to the equation to produce an identity, but were happy not to have an explicit equals sign, concentrated on the solving-for-answer aspect, and tend to have the embodied input-output perspective:

> Q12. Mathematics problem to be solved to get an answer
>
> Q30. Question written numerically and requires an answer, has variables sometimes

Finally, the 4 students in category 5, who were similar to those in category 4, but demanded an explicit equals sign, also emphasised the embodied solution

aspects:

Q20. Mathematics way to calculate or figure out something

Q31. Mathematics statement to find a solution to a problem

Q44. Method of mathematics to find unknown value

Group R

This group is clearly important in our consideration of the changing ideas of equation in the transition from school to tertiary study. One would hope that the undergraduate group might have consolidated their understanding of equation in the symbolic world and be moving towards the acceptance of properties in their own right, as a pre-cursor to formal world thinking. Figure 9 shows some of the questions that were given to the Group R first year university students. Again the examples used in Question 5 did not correspond exactly with those given to the other two groups, but they were intended to be parallel types appropriate for these older students. For example, 5c) is an identity appropriate to this level of student and replaces the identity $5t - t = 4t$ given to the other students. A second equation describing a property, this time the symmetric property has also been added for these students to see if they were developing a property perspective for equivalence.

1. Explain what an equation is. [read question 3 below before answering]
2. Give an example(s) of an equation.
3. What is an equation for?
5. Look at the following list. Decide which ones you consider are equations and circle the Y. Circle the N if you think they are not equations. If you would like to comment please do so beside each statement.

a) $a = 5$ b) $7w - w$ c) $x(x + 2) = x^2 + 2x$ d) $0 = x^2 + 2x - 5$

e) $a = a$ f) $a + b = b + a$

Figure 9. Four questions from the group R questionnaire (some minor changes).

Hence, taking parts a) to d) for the categories we have seen before we would expect a category 1 student who judges on the equals sign to reject b) and answer yes to a), c) and d). A category 2 student who wants an operation to carry out would reject a) and accept b), c) and d), and so on (see Table 7).

This shows that most of the university students are happy simply to require the explicit presence of the equals sign, although there are still a few (3) who place more emphasis on having an operation to carry out. For example student R12 describes an equation as:

> …when two statements are equate[d] with the use of an "=" sign. One of the statement[s] could be the process involved to produce a conclusion and the other one is the conclusion. It could also be two statements equating the different process that come with the same conclusion.

The multiple use of the words 'process' and 'conclusion' in this explanation strongly suggest an emphasis on operational or procedural thinking with a subsequent result or output.

Table 7
Categories of Undergraduate Group R Responses for Question 5 (N=30)

Statements that are an equation	Stated reason for the choice	Number of students in category
a, c, d only	Needs an = sign	17 (58.6%)
b, c, d only	Needs an operation to carry out	3 (10.3%)
c, d only	Needs an = sign and an operation to carry out	4 (13.8%)
b, d only	Needs an operation but is not an identity or an assignment	0
d only	Needs an operation and an = sign but is not an identity or an assignment	0

The other 5 students either had no response or a combination difficult to categorise.

When we examined other responses to Question 1 for an understanding of what an equation is, there was no unanimity of answer; in fact the variety of responses was somewhat surprising. Some students mentioned embodied constructs, such as the input-output idea seen in younger students (eg R6 "An equation is a set of principals [sic] that operate on a given input variable in such a way that gives a different output value"), or focussed on the visual nature of the '=' sign (R2 "Equation is a mathematical representation of a set of variables and constants on either side of an '=' sign"; R20 "Right hand side = Left hand side"). Six students put an emphasis on the process of solving or evaluating to find the value of an unknown that is at the centre of much early work in the symbolic world of the secondary school (R7 "An equation is a set of expressions that have equal value when evaluated at the left hand side and the right hand side in comparison"; R14 "A mixture of numerical and/or alphabetical components which are constructed to assist in problem solving…they usually have unknown values which we try to find"; R15 "An equation is a mathematical formula form by some unknown variables and numbers. And it is those unknown variables we are trying to find a value/answer to it").

Some of the more interesting responses to the question were found in the five students who mentioned relationships as the core of equation (e.g., R9 "Provides a mathematical relationship between 2 or more variables, or gives information about any variable"; R17 "Something used to mathematically describe what is happening in a system and how the different components of it are related to each other"). This last comment shows the influence of engineering science study on their perception.

A number of the student responses could not be easily categorised since they crossed boundaries. For example, R22 described an equation as a "Statement given to solve unknown variables in order to equate the right hand side is equal to the left hand side". This uses the idea of a statement property but then places the emphasis on solving to find an unknown, hence straddling statement and evaluation categories. Further, R21's comment that it is a "Statement of relationship between sets of terms (which is built by one or more mathematical

operations)" crosses between the categories of statement and relationships. R1 was one of only two students who used the word equivalence, saying it is "The way to express equivalence between things".

One of the aspects of equality that undergraduate students need to learn as part of the transition from school is the concept of an equivalence relation. Hence, this study considered the extent to which early undergraduates have this conception. We see from Table 8 that Group R students have made some progress over the school students in understanding the reflexive property, with 18 (60%) accepted $a=a$ as an equation in question 5e), compared with 35.5% of Group Q. However, this indicates that 40% were still forming an understanding of the reflexive property of equivalence relations, or at least accepting its written algebraic representation as an equation. An indication that thinking is still more in the symbolic world than the formal world for some students was the fact that in question 5f) 8, or 26.7%, did not see $a + b = b + a$ as an equation describing the commutativity property of addition, that becomes an axiom in formal world thinking (although 22, or 73.3%, did).

Table 8
First Year Undergraduate Questionnaire Responses

Question 5	Yes	No	No Response
a) $a = 5$	21	8	1
b) $7w - w$	4	26	0
c) $x(x + 2) = x^2 + 2x$	27	2	1
d) $0 = x^2 + 2x - 5$	29	1	0
e) $a = a$	18	12	0
f) $a + b = b + a$	22	8	0

Group Comparisons

When we analyse and compare the results from the Groups P and Q students what do we see? It seems clear that there is a strong sense of procedural necessity that is attached to the concept of equation for these school students, whose thinking is dominated by the symbolic world, and for a significant minority this requirement is strong enough to overrule the need for an explicit '=' sign to be present. The other large group present in both years comprises those who are generally satisfied if the '=' sign is present, but who also require some operation to carry out. These students do not accept a simple assignment as an equation.

The answers to a typical question from a university linear algebra examination raised an issue regarding the transitive property.

(a) The augmented matrix of the linear system

$$\begin{aligned} x + 2y + kz &= 0 \\ 2x + 3y - 2z &= k \\ kx + y + z &= 3 \end{aligned}$$

can be transformed into the matrix $\begin{bmatrix} 1 & 2 & k & 0 \\ 0 & 1 & 2-2k & -k \\ 0 & 0 & 3k^2+2k-1 & 3+k-2k^2 \end{bmatrix}$.

For what values of k (if any) does the system have:

(i) no solutions;

(ii) infinitely many solutions;

(iii) a unique solution.

Figure 10. An undergraduate linear algebra question.

What was interesting in this question (see Figure 10) was that, in part (ii), students have to consider when both quadratic expressions in k are zero. We noticed that some students used the transitive property to argue, correctly, that since we require both $3k^2 + 2k - 1 = 0$ and $3 + k - 2k^2 = 0$ then it must be true that $3k^2 + 2k - 1 = 3 + k - 2k^2$. However, they then went on to accept both solutions of this equation as belonging to the intersection of the solution set of the first two equations. This is an over-generalisation, reversing the implication of the transitive property, and we decided to investigate this further to see how common it was. Question 8 in the questionnaire given to both Groups Q and R used exactly these expressions, asking:

> 8. Sarah argues that if $3k^2 + 2k - 1 = 0$ and $3 + k - 2k^2 = 0$ then it is true that $3k^2 + 2k - 1 = 3 + k - 2k^2$. Is she right? Explain your answer.

For Group Q 33 students (43.4%) said that this was acceptable, while 23 (30.3%) said that it wasn't, and 20 (26.3%) gave no response. Of the 30 students in Group R, 13 (43.3%) answered 'yes', 8 (26.7%) said 'no', 5 (16.7%) that 'it depends', while 4 (13.3%) gave no response. Thus, for both groups, while around 43.3%, agreed that it was correct to equate the two expressions, 27-30% thought that it was not. The reasons given for each position are important, and typical reasons from Group R students for and against using transitivity can be seen in Figures 11 and 12.

We see that most of those who answered that the two expressions were equal reasoned using the transitive property; since both equalled zero then they must equal each other. They are moving towards formal world thinking based on axiomatic properties, and correctly followed reasoning that has been around since Euclid's time, when he wrote as the first of his Axioms "1. Things which are equal to the same thing are equal to one another." (Euclid, 1933, p. 6). Only R21 seems to have addressed the solution issue, accepting the transitivity implication but noting that they simply share "a common solution", not all solutions.

Student	Answer
R1	Because both equation is equivalent to the same values, so one equation must equal to the other
R7	She is right. Both $3k^2+2k-1$ and $3+k-2k^2$ has the same value and that is 0. This ~~Same properties~~ characteristic & of both expression links them to be equal to each other.
R23	Yes, since both of the equations equal to zero, they must be equal to each other.
R21	$3k^2+2k-1=0$ and $3+k-2k^2$ share a common solution i.e. $2k^2+2k-1 = 3+k-2k^2 = 0$

Figure 11. Undergraduate Group R reasons for equating two quadratic expressions.

One has to be careful when applying transitivity to *conditional* equations (Borowski & Borwein, 1989), true for only some values of the variables, not to reverse the implication. This often does not arise at school since the most common use of the transitive property at that level is in identical equations, for objects equivalent for all variable values, such as in:

$$(a+b)^2 = (a+b)(a+b), \qquad \text{and} \qquad (a+b)(a+b) = a^2 + 2ab + b^2$$

$\Rightarrow (a+b)^2 = a^2 + 2ab + b^2$. However, while $f(x) = g(x)$ (conditional) and $g(x) = h(x)$ (conditional) imply the conditional statement $f(x) = h(x)$, $f(x) = h(x)$ does not imply both $f(x) = g(x)$ and $g(x) = h(x)$. Hence, the values of x for which $f(x) = h(x)$ is true will not all necessarily satisfy the first two conditional equations. Only those values in the intersection of the solution sets of the first two equations will.

The students who said that the expressions were not equal gave various reasons and Figure 11 shows some responses. Here students R11, R12 and R29 use reasoning based on finding solutions for k to the two equations separately to identify a potential problem, but did not reconcile this with the notion of transitivity. These students are mainly thinking in the symbolic world, relying on results of manipulations, with student R15 unwilling to accept the transitivity.

When we combine this data together it seems to support the view that these undergraduate students are moving towards formal world thinking about equality that mathematicians espouse, but the subtleties of such reasoning mean that a significant proportion of them still has some way to go to develop it fully.

Student	Answer
R11	No, 1 and 2 is two different equation when solving them seperately, the value of k ~~whit~~ will be different for each equation
R12	NO, because the value of k in the first equation is not equal to the value of k found in the ~~seqt~~ second equation.
R29	$k_1 = -1$ $k_2 = \frac{1}{3}$ $k = \frac{1}{3}$ $k_2 = -3$ NO
R15	NO, even though bothe equations has a value equal to 0, but it doesnot necessarily mean that the two equation are equal to each other.

Figure 12. Undergraduate Group R reasons for not equating quadratic expressions.

Discussion

Consideration of the nature of object construction in mathematics (eg Tall, Thomas, Davis, Gray, Simpson, 2000) is useful when analysing the use of equations in mathematics and the transition from embodied and symbolic thinking to formal world thinking. As Tall et al. observe, some mathematical objects are purely theoretical, or constructs from formal thinking, with no physical counterpart, while others do have such a counterpart. For example, Fischbein (1993, p. 141) refers to how "successful geometric reasoning can be achieved when we stop considering only two distinct categories of mental entities (images and concepts) and we deal apart from them with a third type of mental object, the figural concept". This idea of a *figural concept* in geometry has also been examined by Laborde (2002), who drew a distinction between drawings and figures, explaining the former as physical and perceptual, and latter as theoretical and mathematical. To use semiotic language we may say that the perceived object changes roles from an icon to a symbol. According to Thomas (2006), this change in interpretation of the external sign (or representation) involves a link to an appropriate, existing mathematical schema to ascertain the properties that may be overlaid in memory on the external sign (see Figure 13).

Thus, for example, the schema for the mathematical concept of rectangle, is a combination of a perception of an embodied sign or external representation, its object referent and mathematical data (including properties). It is addition of the properties that constitute the mathematical object of rectangle (enabling a decision

on what makes an object 'not a rectangle'), namely two pairs of opposite sides equal and four 90° angles, to a schema for rectangle, that we have begun to construct the mathematical concept. On the other hand, a pseudo-conception (Vinner, 1997) of rectangle can lead to errors, such as the common one, based on perception alone, that a square is not a rectangle. The lack of a well-formed rectangle schema prevents reasoning that it satisfies the required properties. Thus the ability to see squares as a subset of rectangles is due to seeing different mathematical objects and the relationship between their mathematical properties.

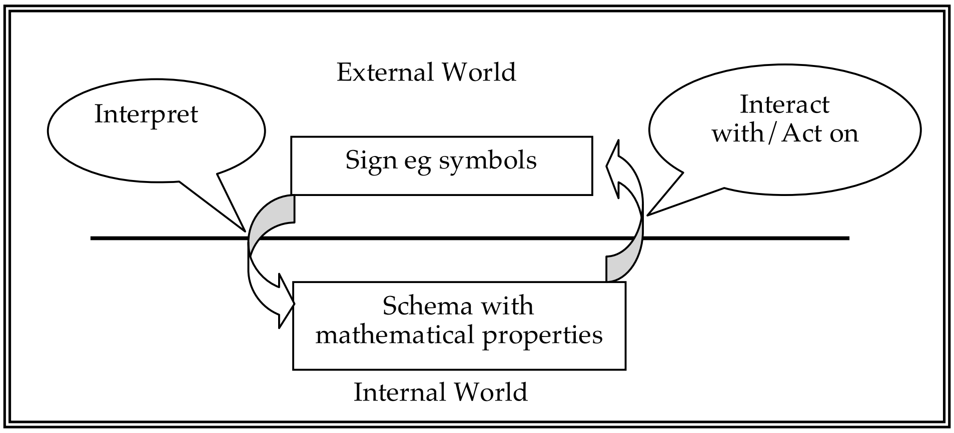


Figure 13. 'Appropriate' mathematical schema used to interact with external sign.

However, these ideas need not be limited to geometric constructions, since most mathematical objects have symbolisms that may be viewed in different ways. A procept is a symbolic world object having possible variations on perspective, the process versus object view of the symbol. But this is not the only duality for symbols. When we perceive the symbolisation of an object we may simply have a surface or observational view (Thomas, 2001), but in order to get a mathematical perspective of what it represents we have to interpret what we see using our mathematical schemas (c.f., Booth & Thomas, 2000; Thomas, 2006). This process of interpretation of the symbolisation, or representation, may require us to interact with the figural-concept object (Thomas & Hong, 2001), giving rise to identification of the object's properties, often underlying operational invariants (c.f. Vergnaud, 1998).

Considering the situation with regard to the equation concept, with its '=' sign, it appears that organising the properties that define the mathematical object into a coherent schema can be elusive for students. Problems generalising properties appear to start early, with Warren (2001), for example, showing that many Year 4 students had already developed misunderstandings with regard to the commutative property in arithmetic equations. As they progress from mainly embodied world thinking, through that of the symbolic world, to formal thinking, their perspective on equation changes gradually. Our evidence is that an embodied input-output, or a procedural/operational view of equation is quite persistent for around 25% of students, even during transition to university. Some of these form a recognition of an equation based on the surface observation that it contains an '=' sign, as a number of students in this study did, but they still require the

operational aspect. An increasing proportion of our students focused solely on the presence of the '=' sign, with 27.6%, 31.6% and 58.6% of group P, Q and R students respectively in this category.

We have seen that the appreciation of properties develops slowly. While 73.3% of university students were willing to see $a + b = b + a$ as an equation, only 35.5% of group Q students accepted the reflexive $a=a$ as an equation, along with 60% of the university students. While 62.5% of the younger, group P students tackled an easy problem based on the transitive property, we cannot be sure that they did not resort to symbolic world manipulation in order to answer it. We have also identified a difficulty due to reversing the implication of this property. Thus even for students in the first year of university, understanding of the use of equality is often not predicated on explicit construction of properties of equations, such as the reflexive, symmetric and transitive nature of the '=' sign, and the use of letter as variable, but is still based in symbolic world thinking.

We suggest that, since the mathematical equation object comprises a number of disparate symbols for: arithmetic numbers; variables; operators; and the equals sign; along with formal properties of an equivalence relation, and the structure combining them, understanding the gestalt object requires multiple layers of schematic properties to be overlaid on the iconic view. This structural view is no doubt fed by a developing understanding of properties of the constituent parts, and in turn feeds back to further understanding of the object. One's mathematical understanding of these parts eventually becomes welded into a more or less coherent, schematic whole, with embedded sub-schemas for each part, making the mathematical equation object much greater than the sum of its parts. The evidence we have is that students may have quite well-developed understanding of one or more of the constituent parts, such as the role of operators, but have less comprehension of others.

We have concentrated in this paper on what students describe an equation to be, how they decide what is, or is not an equation, and their ability to recognise and use the symmetric, reflexive and transitivity properties of equivalence. We have distinguished a variety of perspectives that contribute to a view of equation and these are presented in our outline framework for equation (Figure 14). Of course, all our mathematical understanding is mediated by language and so any framework must include its vital role, and we plan to investigate the binding influence of this, and the role of a structural perspective.

It appears that one reason why students do not construct the properties of an equals sign as an equivalence relation is that teachers at school, and first year university, often use the symmetric, reflexive or transitive properties of equals without making these explicit. For example, when solving an equation we may go from $x + 6 = 3x + 1$ to $2x + 1 = 6$, rather than $6 = 2x + 1$, using the symmetric property applied to the *conditional* equation. Or we may reason along the lines that if $y = 2x + 1$ (*identical* equation, defining y), then when $y = 0$ (*conditional* equation), $2x + 1 = 0$ (*conditional* equation), employing the transitive property to do so. However, we may not explicitly highlight these properties, or the kinds of equations employed, leaving students to abstract these themselves. If students have a view of the equals sign as signifying the result of a procedure, or only as a conditional equality, as many in our study did, and have not constructed the properties of an equivalence relation, they will not be able to interact fully with the mathematical equation object, or make good progress in their transition to university mathematics.

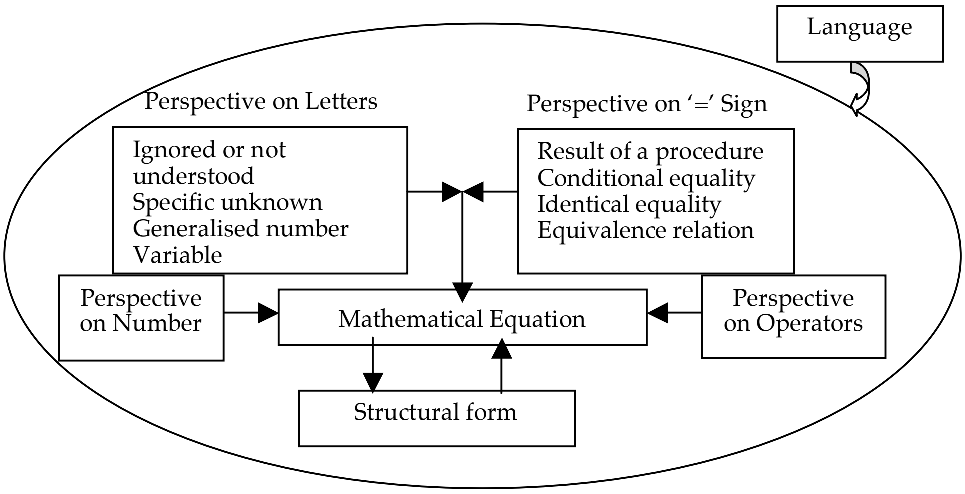


Figure 14. An outline framework of the mathematical equation object.

One conclusion is that, since the equation object is fundamental to mathematical understanding, teachers should make a deliberate effort to assist students to enrich their perspective on equation by paying explicit attention to the structure and properties of equations, allowing students to interact fully with the mathematical equation object and hence construct a rich schema. Jones and Pratt (2006) report on the value to 13 year-old students of technology-supported utilities for the equals sign, and their changing conceptions of the sign as a result. It may be that interaction with powerful computer-based CAS systems that employ different uses of the equals sign in different mathematical equation contexts might be a way to assist older students to construct such a rich equation schema. In this manner the transition from school to university may be made a little smoother.

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